MULTIFRACTAL STRUCTURE AND

PRODUCT OF MATRICES*

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Abstract

There is a well established multifractal theory for self-similar measures generated by non-overlapping contractive similutudes. Our report here concerns those with overlaps. In particular we restrict our attention to the important classes of self-similar measures that have matrix representations. The dimension spectra and the L^q-spectra are analyzed through the product of matrices. There are abnormal behaviors on the multifractal structure and they will be discussed in detail.

1 Introduction

We call a family of contractive maps $\{S_j\}_{j=1}^N$ on $\mathbf{R}^d$ an iterated function system (IFS).

It is well known that an IFS will generate an invariant compact subset $K = \bigcup_{j=1}^{N} S_j K$, which

we usually refer to as a fractal set. If further, we associate a set of probability weights

$\{w_j\}_{j=1}^N$ to the IFS, then it will generate an invariant measure

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$$\mu = \sum_{i=1}^{N} w_j \mu \circ S_j^{-1}.$$ (1.1)

It has (Borel) support contained in K and is dense in K. The invariant sets and measures play a central role in the theory of fractals. The above contractions are, however, too general to yield concrete results. Therefore we often assume that the maps are similitudes (μ is called a self-similar measure), and in the iteration process, they satisfy a nonoverlapping condition called the open set condition (OSC) [Hut].

If the iteration process has overlap, the situation is more complicated and it has attracted a lot of interest recently. The simplest IFS that has overlap is the maps

$$S_1 x = \rho x, \quad S_2 x = \rho x + 1, \quad x \in \mathbf{R}$$ (1.2)

with $1/2 < \rho < 1$. The invariant measure μ_ρ with weights $1/2$ on each map is called the Bernoulli convolution. It is the distribution of the sum $\sum_{n=0}^{\infty} \rho^n \mathbf{X}_n$ where $\{\mathbf{X}\}_{n=0}^{\infty}$ is a sequence of i. i. d. random variables taking values $\{0, 1\}$ with probability $1/2$ each. For the special case of golden ratio $\rho = (\sqrt{5} - 1)/2$, we call μ_ρ the Erdös measure. There are elegant results on the family of Bernoulli convolutions and they have inspired the development of the fractal theory considerably. The reader can refer to [PSS] for a nice survey of the development. But as a whole, the iteration by the maps in (1.2) is still not completely understood.

In [LN2], Ngai and the author introduced a weak separation condition (WSC) on the IFS of simililtudes. The condition is weaker than the OSC and includes many of the important overlapping cases (see the examples in Section 3). In particular it includes the Erdös measure (or more generally, the Bernoulli convolution with ρ^{-1} a P. V. numbers). This new condition also applies to the scaling function in wavelets (e. g. [D], [DL1], [LWa]). There are two basic questions concerning the self-similar measures:

1. The absolute continuity or singularity of μ.

2. The multifractal structure of μ.

For the first question, it is known that μ is either continuously singular or absolutely continuous with respect to the Lebesgue measure. By assuming the WSC, Ngai, Rao and the author[LNR] gave a necessary and sufficient condition on the absolute continuity of μ in term of the weights w_j. The condition is further extended to absolute continuity with respect to the Hausdorff measures[LW1].

The purpose of the paper is to report on some recent work on the second question un-

der the WSC. The measures under consideration are necessarily singular (for the absolutely continuous measure, one can consider the corresponding regularity property of the Radon-Nikodym derivative of μ instead). The goal is to prove the multifractal formalism of the dimension spectrum $f(\alpha)$ and the L^q-spectrum $\tau(q)$ (defined in Section 2). In [LN2], the formalism was proved valid at the point where $\tau(q)$, $q>0$, is differentiability of $\tau(q)$. The main question is

3. Is $\tau(q)$, $q \in \mathbf{R}$, differentiable? What about the multifractal formalism for $q<0$?

From the definition, it is easy to show that $\tau(q)$ is a concave function and hence differentiable except for at most countably many points. In [LN1], Ngai and the author studied the Erdös measure and obtained a formula for $\tau(q)$, $q>0$; the formula implies the differentiability of $\tau(q)$, $q>0$. The technique is to reduce the self-similar identity (1.1) to a set of vector identities with a new "nonoverlapping" IFS [STZ]; the corresponding probability weights are then reduced to a set of matrices. The formula was derived by using the product of these special matrices. Later Feng[F1] extended the formula to $q \leqslant 0$ and found that $\tau(q)$ has a non-differentiable point at $q<0$. This is a striking result, nevertheless the multifractal formalism still holds [FO].

Another instructive example of WSC is the three—fold convolution μ of the Cantor measure. By using a matrix representation and applying a similar analysis as the previous case, Wang and the author[LW] obtained a formula for $\tau(q)$ with a non-differentiable point as for the Erdös measure. The more interesting results is that the set of local dimensions has an isolated point (Hu and Lau[HL]), it is in contrary to all the previously known cases of IFS that this set should be an interval. The multifractal formalism has to be modified to adjust for the isolated point [FLW]. In a detail study, Shmerkin[S] extended this example considerably and gave a necessary and sufficient condition for the existence of isolated points in the set of local dimensions.

The expression of a self-similar measure into product of matrices can be applied to a larger class of IFS. Actually this approach has been used in the scaling functions in wavelet theory (where the weights are real coefficients instead), and the product of the matrices was used to study the existence of the L^p-scaling functions and their regularity[DL2], [LMW]. Furthermore Feng[F2] proved for the class of IFS on $\mathbf{R}$ satisfying afinite type condition (which implies WSC), the self-similar measure can be expressed as product of matrices locally.

The product of matrices in connection with the differentiability of $\tau(q)$ was investigates by Feng and the author[FL] in a more general setting. Let $\{M_1, \cdots, M_N\}$ be non-negative matrices. For $J = (j_1, \cdots, j_n)$, let $|J|$ be the length of J and let $M_J = M_{j_1} \cdots M_{j_n}$.

Let

$$P(q) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|J|=n} \| M_J \|^q, \tag{1.3}$$

where $\| M_J \| = \mathbf{1} M_J \mathbf{1}^t$ where $\mathbf{1} = [1, \cdots, 1]$. Under some conditions on the M_i's, they showed that $P(q)$, $q > 0$ is differentiable. For the special class of self-similar measures satisfying the finite type condition, $P(q)$ differs from $\tau(q)$ only by a factor, this implies that $\tau(q)$, $q > 0$ is differentiable and by the result of [LN2], the multifractal formalism is valid for $q > 0$.

In Section 2, we will first consider the contractive similitudes with the OSC and discuss certain background material and the well known results related to the dimensions. We introduce the WSC and report some basic theorems in Section 3. The recent development of the finite type condition, the matrix representation of a self-similar measure, the product of matrices and the multifractal formalism are discussed in Section 4. Finally in Section 5 and 6, we use the Erdös measure and the convolution of the Bernoulli convolutions as concrete examples to explain the theory as well as the possible developments.

2   Preliminaries

Throughout we assume that $\{S_j\}_{j=1}^N$ is a finite family of contractive similitudes on $\mathbf{R}^d$, i. e. ,

$$S_j x = \rho_j R_j x + b_j,$$

where $0 < \rho_j < 1$, R_j is an orthonormal matrix, and $b_j \in \mathbf{R}^d$. The invariant measure in (1.1) is called a self-similar measure. We say that $\{S_j\}_{j=1}^N$ satisfies the open set condition (OSC) if there exists a bounded nonempty open set U such that

$$S_i(U) \subseteq U \quad \text{and} \quad S_i(U) \cap S_j(U) = \varnothing, \quad \forall \, i \neq j.$$

The "singularity" of a measure is studied through the various notions of dimension. The most basic one is the local dimension of μ at x defined as

$$\alpha(x) = \lim_{h \to 0^+} \frac{\log \mu(B_h(x))}{\log h}, \qquad x \in \mathrm{supp}\mu \tag{2.1}$$

provided the limit exists where $B_h(x)$ is the ball centered at x with radius h. This means

that $\mu(B_h(x)) \approx h^{a(x)}$ as h is sufficiently small. The local dimension is quite difficult to handle in general. In the sequel we see that it is more fruitful to consider through the global dimensions of μ. We define the Hausdorff dimension of μ as

$$\dim_{\mathscr{H}}(\mu) = \inf\{\dim_{\mathscr{H}}E : \mu(\mathbf{R}^d \backslash E) = 0\},$$

and the entropy dimension (also called the entropy) of μ as

$$\dim_1(\mu) = \lim_{h \to 0^+} \frac{\inf \sum_i \mu(B_h(x_i))\log\mu(B_h(x_i))}{\log h}$$

where $\{B_h(x_i)\}_i$ is a disjoint family of balls, and the infimum is taken over all such families (assume $0 \cdot \infty = 0$). The following theorem is well known [Y].

Theorem 2.1. *Suppose μ is a probability measure on $\mathbf{R}^d$ such that the local dimension of x equals α for μ-almost all x, then $\dim_{\mathscr{H}}(\mu)$ and $\dim_1(\mu)$ are both equal to α.*

For more detail of the relationship of these dimensions, the reader can refer to [N], [Heu] and [FaL]. For the self-similar measure with the OSC, the α in the above theorem is explicit[GH], [CM], [St]:

Theorem 2.2. *Let μ be a self-similar measure defined by $\{S_j\}_{j=1}^N$. Suppose $\{S_j\}_{j=1}^N$ satisfies the OSC. Then for μ-almost all x the local dimension of μ is*

$$\alpha_1 = \sum_{j=1}^N w_j\log w_j \bigg/ \sum_{j=1}^N w_j\log\rho_j.$$

Another important notion of the dimension of a measure is the L^q-scaling spectrum (or L^q-spectrum for short) $\tau(q)$. We define

$$\tau(q) = \lim_{h \to 0^+} \frac{\log \sup_i\left\{\sum_i \mu(B_h(x_i))^q\right\}}{\log h}, \quad q \in \mathbf{R}. \tag{2.2}$$

where the supremum is taken over all disjoint family of closed h-balls $\{B_h(x_i)\}_i$ with $x_i \in$ suppμ. It is easy to show that $\tau: \mathbf{R} \to [-\infty, \infty)$ is an increasing concave function with $\tau(1) = 0$. Note also that $\tau(0)$ is the lower box dimension of suppμ, and formally $\dim_1(\mu)$ is $\tau'(1)$ when both limits exist.

For $q \geqslant 0$ we can replace the packing of disjoint balls in the definition of $\tau(q)$ by the h-mesh cubes $\{Q_h(x_i)\}_i$ that intersect suppμ:

$$\tau(q) = \lim_{h \to 0^+} \frac{\log \sum_i \mu(Q_h(x_i))^q}{\log h}, \quad q > 0.$$

The equality is false for $q < 0$, but it can be adjusted by replacing $Q_h(x_i)$ with the new cubes $\widetilde{Q}_h(x_i)$, which are three times larger and have the same centers x_i[R]. A more complete analysis are given in [BGT] and [GT]. In [PS2], Peres and Solomyak proved.

Proposition 2. 3.   *Let μ be a self-similar measure defined by the contractive simili-tudes $\{S_j\}_{j=1}^N$, then the limit in the definition of $\tau(q)$, $q>0$, exists.*

With the OSC on the $\{S_j\}_{j=1}^N$, $\tau(q)$ has an explicit expression[CM].

Theorem 2. 4.   *Let μ be a self-similar measure defined by the contractive similitudes $\{S_j\}_{j=1}^N$ that satisfies the OSC. Then the L^q-spectrum $\tau(q)$ is given by*

$$\sum_{j=1}^N w_j^q \rho_j^{-\tau(q)} = 1, \qquad q \in \mathbf{R}. \tag{2.3}$$

To return to the local dimension $\alpha(x)$ defined in (2.1), we let $K_\alpha = \{x: \alpha(x) = \alpha\}$ and let

$$f(\alpha) = \dim_{\mathscr{H}} K_\alpha.$$

We call $f(\alpha)$ the dimension spectrum (also called the singularity spectrum) and refer to μ as a multifractal measure if $f(\alpha) \neq 0$ for a continuum of α. This concept was first proposed by the physicists to study various multifractal models arising from the consideration of turbulence (e. g. , Mandelbrot[M], Hentschel and Procaccia[HP], Frisch and Parisi[FP], Halsey et al[HJ]). In order to determine $f(\alpha)$, they proposed to use the L^p-spectrum and the Legendre transformation based on some physical intuition: The function $\tau: \mathbf{R} \rightarrow [-\infty, \infty)$ is a lower semi-continuousconcave function. We denote the effective domain of τ by Dom $\tau = \{q: -\infty < \tau(q) < \infty\}$. The Legendre transformation (concave conjugate) of τ is defined by

$$\tau^*(\alpha) = \inf\{\alpha q - \tau(q): q \in \mathbf{R}\}.$$

If τ is differentiable at q, then $\alpha = \tau'(q)$ and

$$\tau^*(\alpha) = \tau'(q)q - \tau(q).$$

By using some heuristic arguments, the physicists[HJ],[HP],[FP] suggested that the following relationship holds

$$f(\alpha) = \tau^*(\alpha).$$

It is called the multifractal formalism. In many respects, it is analogous to the well known thermodynamic formalism[Ru]. The following proposition can be proved using the Vitali-covering theorem (see e. g. , [Fa], [LN2]).

Proposition 2. 5.   *Let μ be a probability measure and let τ be the L^q-spectrum , then for $\alpha \in (\mathrm{Dom}\tau^*)^\circ$, $f(\alpha) \leqslant \tau^*(\alpha)$.*

However the reverse inequality is much harder to obtain. It depends on the IFS and so far, no complete answer is in sight. The following is a complete answer for the contractive similitudes satisfying the OSC[CM], [LN2].

Theorem 2. 6.   *Let μ be an invariant measure defined by a contractive IFS $\{S_j\}_{j=1}^N$ of similitudes with the OSC, then $f(\alpha) = \tau^*(\alpha)$ for $\alpha \in (\mathrm{Dom}\tau^*)^\circ$.*

The proof makes used of the Birkhoff ergodic theorem and Theorem 2. 4. We see that $\tau(q)$ is differentiable and the derivative is

$$\alpha(=\alpha(q)) = \tau'(q) = \frac{\displaystyle\sum_{j=1}^{N}(\log w_j)w_j^q\rho_j^{-\tau(q)}}{\displaystyle\sum_{j=1}^{N}(\log\rho_j)w_j^q\rho_j^{-\tau(q)}}, \quad q \in \mathbf{R},$$

and $f(\alpha)=\tau^*(\alpha)=\alpha q-\tau(q)$. Note also that $\tau'(1)$ is the α_1 in Theorem 2. 2.

3   The Weak Separation Condition

Our goal in this section is to relax the OSC to admit many important IFS with over-laps. The main idea is to use points separation instead of the bounded open set in the OSC.

Let $\{S_j\}_{n=1}^{N}$ be contractive similitudes. For $J=(j_1,\cdots,j_n)$, $1\leqslant N$, we let $|J|$ denote the length of J, $S_J=S_{j_1}\cdots S_{j_n}$, $\rho_J=\rho_{j_1}\cdots\rho_{j_n}$. For $r>0$, we let

$$\mathcal{J}_r = \{J = (j_1,\cdots,j_n) : \rho_J \leqslant r < \rho_{j_1\cdots j_{n-1}}\}.$$

Note that S_J may equal $S_{J'}$ even for $J\neq J'$. The following definition was given in [LN2].

Definition 3. 1.   An IFS of contractive similitudes $\{S_j\}_{j=1}^{N}$ is said to satisfy the weak separation condition (WSC) if there exists $x_0\in\mathbf{R}^d$ and a constant $a>0$ such that for any $r>0$ and for any J, $J'\in\mathcal{J}_r$,

$$\text{either} \quad S_J(x_0) = S_{J'}(x_0) \quad \text{or} \quad |S_J(x_0) - S_{J'}(x_0)| \geqslant ar. \tag{3. 1}$$

The definition says that in the iteration, if the $S_J's$ have (approximatively) the same contracting ratio r, then all the states $S_J(x_0)$ are either identical or separated by a distance ar. It is easy to see that (3. 1) is equivalent to:

$$\text{either} \quad S_J^{-1}S_{J'}(x_0) = x_0 \quad \text{or} \quad |S_J^{-1}S_{J'}(x_0) - x_0| \geqslant a', \quad \forall J, J' \in \mathcal{J}_r. \tag{3. 2}$$

In [BG] Bandt and Graf showed that $\{S_j\}_{j=1}^{N}$ satisfies the open set condition if and only if there exists $x_0\in\mathbf{R}^d$ and $a>0$ such that $|S_J^{-1}S_{J'}(x_0)-x_0|\geqslant a'$ for all incomparable J and J'. It follows that

Proposition 3. 2.   *If $\{S_j\}_{j=1}^{N}$ satisfies the OSC, then it satisfies the WSC.*

The following are some useful equivalent conditions for the WSC[LW1], [Z].

Proposition 3. 3.   *Let $\{S_j\}_{j=1}^{N}$ be contractive similitudes, then the following are equivalent:*

(i) $\{S_j\}_{j=1}^{N}$ *satisfies the WSC;*

(ii) *any ball $B_r(x)$ contains at most $l(\leqslant [2a^{-1}]^d)$ of the $S_J(x_0)$;*

(iii) *there exists $\gamma > 0$ with a compact subset $D \subset \mathbf{R}^d$ and $\bigcup_{j=1}^{N} S_j(D) \subset D$ such that for any $r > 0$ and $x \in \mathbf{R}^d$,*

$$\# \{S \in \mathscr{A}_r : x \in S(D)\} \leqslant \gamma,$$

where $\mathscr{A}_r = \{S : S = S_J, J \in \mathscr{J}_r\}$.

Example 3. 1.   Let $\{S_j\}_{j=1}^{N}$ be defined on $\mathbf{R}$ such that $S_j x = \dfrac{1}{k} (x + b_j)$ where $k \geqslant 2$ is an integer, and $b_j = cr_j$ with $c \in \mathbf{R}$ and r_j rationals.

This is one of the most important class of IFS. The case $k = 2$ has been studied in great detail in wavelet theory. We will discuss this in Section 4. For $k = 3$, the most well known case relates to the Cantor measure μ where $b_0 = 0$, $b_1 = 2$ which satisfies the OSC; the three-fold convolution μ corresponds to $S_j(x) = \dfrac{1}{3}(x + b_j)$ with $b_i = 0$, 2, 4, 6, this IFS has the WSC but not the OSC. The multifractal structure of μ will be discussed in detail in Section 6.

Example 3. 2.   Let $\{S_j\}_{j=1}^{N}$ be defined on $\mathbf{R}$ such that $S_j x = \rho x + b_j$ where $\beta = \rho^{-1}$ is a P. V. number (i. e. , $\beta > 1$ is an algebraic integer such that all its algebraic conjugates have modulus less than 1), and $b_j = c \, r_j$ with $c \in \mathbf{R}$ and r_j rationals. The WSC follows from a lemma of Garsia ([G,Lemma 1. 51], [LNR]).

Besides these standard examples where the contracting ratios are the same and are algebraic numbers, we can also construct examples with more arbitrarily contracting ratios[LNR], [LW2].

Example 3. 3.   Let $0 < \rho < \dfrac{1}{3}$, $S_1 x = \rho x$, $S_2 x = \rho x + \rho$ and $S_3 x = \rho x + 1$, $x \in \mathbf{R}$, then it is easy to prove the WSC as in Example 3. 1. That $S_1 \circ S_3(x) = S_2 \circ S_1(x) = \rho^2 x + \rho$ implies that the OSC is not satisfied by the result of [BG] discussed before Proposition 3. 2.

Example 3. 4.   Let

$$S_1(x) = \rho x, \quad S_2(x) = rx + \rho(1 - r), \quad S_3(x) = rx + (1 - r)$$

with $0 < \rho < 1, 0 < r < 1$ and $\rho + 2r - r\rho \leqslant 1$. Then, with some work, $\{S_j\}_{j=1}^{3}$ satisfies the WSC.

There are also interesting examples in $\mathbf{R}^d$.

Example 3. 5.   On $\mathbf{R}^d$, we let $S_j x = A(x + d_j)$, $j = 1, \cdots, N$ where $B = A^{-1}$ is an integral expanding self-similar matrix, and $d_i \in \mathbf{Z}^d$ with $d_1 = 0$. The WSC is an easy consequence of the definition.

This class of IFS has been studied in detail in connection with the theory of tiles (assuming that $N=\det B$) and also in the context of multivariate scaling functions.

Example 3.6. Let A be the self-similar matrix as in Examples 3.5. and let Γ be a finite group of integral matrices γ with $\det\gamma=\pm1$ and satisfies $\Gamma A=A \Gamma$. Let $S_jx=A_j(x+d_j)$ where $A_j=\gamma_j A, \gamma_j\in\Gamma$, $d_j\in\mathbf{Z}^d$. Then similar to the above example it is easy to show that the IFS satisfies the WSC. For example let

$$A^{-1}=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}, \quad \Gamma=\left\{\begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix}, \begin{bmatrix}0 & \varepsilon_1\\ \varepsilon_2 & 0\end{bmatrix}: \varepsilon_i=\pm1\right\}.$$

That $S_1x=Ax, S_2x=\gamma Ax+\begin{bmatrix}1\\0\end{bmatrix}$ with $\gamma=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ generate an invariant set called the Lévydragon.

The following is a complete characterization of the absolute continuity (equivalently, the singularity) of the self-similar measures under the assumption of WSC[LW1].

Theorem. *Let $\{S_j\}_{j=1}^N$ be contractive similitudes satisfying the WSC. Let K be the self-similar set with $\dim_{\mathscr{H}} K=\alpha$ and let μ be a self-similar measure defined by $\{S_j\}_{j=1}^N$. Then μ is singular with respect to $\mathscr{H}^\alpha|_K$ if and only if there exists $r>0$ and $S\in\mathscr{A}_r$ such that $\rho_S{}^\alpha<w_S$ where $w_S=\sum\{w_J: S_J=S, S_J\in\mathscr{A}_r\}$.*

The more difficult part of the proof is on the sufficiency, the main idea follows from [LNR] for the special case where $\rho_1=\cdots=\rho_N$ and the absolute continuity is for the Lebesgue measure. The theorem offers a very convenient criterion to check the singularity (or absolute continuity) of a self-similar measure under the WSC. In particular for $\alpha=d$, the theorem gives a necessary and sufficient condition for the absolute continuity with respect to the Lebesgue measure.

For the WSC, there is no analogous formula for $\dim_{\mathscr{H}} K$ as is in the case of open set condition (where the dimension is the α satisfies $\sum_j\rho_i{}^\alpha=1$ [Fa]). However we have algorithms to calculate the dimension[HLR], [NW].

As an important consequence of Theorem 1, we have

Corollary 3.5. *Under the above assumption, if the self-similar measure μ is absolutely continuous with respect to $\mathscr{H}^\alpha|_K$, then the Radon-Nikodym derivative of μ is bounded.*

We remark that in Theorem 1, the assumption that $\{S_j\}_{j=1}^N$ has WSC is essential. For considering the Bernoulli convolution μ_ρ as defined in §1, Peres and Solomyak[PS1] proved that if $w\in[\frac{1}{3}, \frac{2}{3}]$, then μ_ρ is absolutely continuous for almost all $\rho\in[w^w(1-w)^{1-w},$

1]. Note that $(\frac{1}{3})^{1/3}(\frac{2}{3})^{2/3}\approx 0.5291<\frac{2}{3}$. Hence if we take ρ and w such that $\frac{2^{2/3}}{3}\leqslant\rho<w$ $=2/3$, we see that there are μ_ρ absolutely continuous with respect to the Lebesgue measure. Theorem 1 does not apply to these μ_ρ.

Furthermore, it was proved in [HLW] that if the $\{S_j\}_{j=1}^N$ satisfies $\rho_j^d<w_j$ for some j and μ is absolutely continuous with respective to the Lebesgue measure (note that $\{S_j\}_{j=1}^N$ cannot have WSC in view of Theorem 1, then the Radon-Nikodym derivative is unbounded. Inparticular the above example of μ_ρ has unbounded derivative.

It is also interesting to know that if the self-similar measure μ is absolutely continuous with respect to $\mathcal{H}^\alpha|_K$, then the two measures must be equivalent, disregarding whether $\{S_j\}_{j=1}^N$ has the WSC or not ([HLW], [MS]).

To conclude this section, we give a theorem on the multifractal formalism[LN2].

Theorem 3.6   *Let $\{S_j\}_{j=1}^N$ be contractive similitudes with the WSC and let μ be a self-similar measure. Suppose $\tau(q)$ is differentiable at some $q>0$, then for $\alpha=\tau'(q)$*

$$f(\alpha):=\dim_H K_\alpha=\tau^*(\alpha).$$

The remaining question in the theorem is to show that $\tau(q), q>0$ is differentiable. This will be considered in the rest of the paper. We do not know whether the theorem is true for $q<0$. In the examples in Section 5,6, we see that there will be non-differentiable point and other abnormalities.

4   Product of Matrices

In this section we will consider expressing the self-similar measures in terms of product of matrices. First let us review an instructive case which has been studied thoroughly in wavelet theory. Consider the following dilation equation

$$\phi(x)=\sum_{j=0}^N c_n\phi(2x-j),\qquad x\in\mathbf{R},\qquad\qquad(4.1)$$

where c_n are real and $\sum_j c_n=2$. The compactly supported L^1-solution $\phi(x)$ is called a scaling function. It is easy to see that formally, the scaling function ϕ in (4.1) corresponds to the Radon-Nikodym derivative of the self-similar measure μ in (1.1) (except the coefficient may be negative) and the IFS is

$$S_j(x)=\frac{1}{2}(x+j),\quad j=0,\cdots,N.$$

Obviously this IFS satisfies the WSC but not the OSC when $N>1$. Note that the support of $\phi(x)$ is contained in $[0, N-1]$ necessarily. We define a vector-valued function on $[0, 1]$ by

$$\phi(x) = [\phi(x), \phi(x+1), \cdots, \phi(x+(N-1)]'$$

and let $M_0 = [c_{2i-j-1}]_{1\leqslant i,j\leqslant N}$ and $M_1 = [c_{2i-j}]_{1\leqslant i,j\leqslant N}$ be the associated matrices of the coefficients $\{c_n\}$, i.e.,

$$M_0 = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_4 & c_3 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{bmatrix}, \quad M_1 = \begin{bmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ c_5 & c_4 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_N \end{bmatrix}.$$

Then (4.1) is equivalent to the following equation

$$\phi(x) = M_0\phi(2x) + M_1\phi(2x-1)$$

([DL1], [LWa]). In this case the corresponding IFS is $T_1(x)=x/2$, $T_2(x)=(x+1)/2$ and satisfies the OSC. The matrix M_0+M_1 has eigenvalue 2, let $\mathbf{v}$ be the left 2-eigenvector. For any $J=(j_1,\cdots, j_n)$, $j_i=0$ or 1, we let

$$\Delta_J = [\sum_{k=1}^{n} j_k/2^k, \sum_{k=1}^{n} j_k/2^k + 1/2^k]$$

and define

$$\phi_n(x) = \sum_{|J|=n} M_J \mathbf{v} \cdot \chi_{\Delta_J}(x).$$

The ϕ_n is used to approximate ϕ; the convergence of the sequence and the regularity of ϕ depend on the joint spectral radius[DL1]

$$\rho(M_0, M_1) = \limsup_{n\to\infty} (\max_{|J|=n} \| M_J|_E \|)^{1/n}$$

and the p-mean spectral radius[J], [LWa]

$$\rho_p(M_0, M_1) = \limsup_{n\to\infty} \frac{1}{2}\Big(\sum_{|J|=n} \| M_J|_E \|^p \Big)^{1/n},$$

where E is the $(d-1)$-dimensional subspace of $\mathbf{R}^d$ orthogonal to $(1,\cdots,1)$ and $M|_E$ means the restriction of the matrix M on E.

To put into our context, we let $\mu(E) = \int_E \phi$, then it is easy to see that

$$\mu(\Delta_J + i) = \frac{1}{2^n}\mathbf{1}'M_J\mathbf{e}_i.$$

The other form of matrix representation was used by Strichartz et al[STZ] and Ngai and

Lau[LN1] to study the Erdös measure. Recently it was generalized by Feng in [F2] to a lager class of IFS.

Definition 4.1.   Let $S_j(x) = \rho x + b_j$, $0 < \rho < 1$, $b_j \in \mathbf{R}$, $j = 1, \cdots, N$. $\{S_j\}_{j=1}^N$ is said to satisfy the finite type condition if there is a finite set Γ such that for each integer $n > 0$, and for any two indices J, J' of length n,

either $\rho^{-n} |S_J(0) - S_{J'}(0)| > c$ or $\rho^{-n} |S_J(0) - S_{J'}(0)| \in \Gamma$,

where $c = (1 - \rho)^{-1} \max\limits_{1 \leqslant i, j \leqslant N} \{b_i - b_j\}$.

This definition is a special case of the more general finite type condition defined by Ngai and Y. Wang[NW] for similitudes on $\mathbf{R}^d$, and it implies the WSC[Ng]; Examples 3.1-3, 3.5-6 all satisfy the finite type condition. This condition yields a graph-directed set on the indices. By some meticulous work on assigning the probability weights to the graph, Feng[F2] proved:

Theorem 4.2.   *Let* $\{S_j\}_{j=1}^N$ *be a co ntractive similitudes on* $\mathbf{R}$ *satisfying the finite type condition, and let* μ *be a self-similar measure. Then there exist families* $\mathscr{F}_n$, $n = 1, 2, \cdots$ *of closed intervals with disjoint interiors, and a set of non-negative square matrices* $M_1, \cdots,$ M_m *such that*

(i) *each* $\Delta \in \mathscr{F}_n$ *is contained in exactly one* $\Delta' \in \mathscr{F}_{n-1}$;

(ii) $K = \bigcap\limits_{n=1}^{\infty} (\cup \mathscr{F}_n)$;

(iii) $H := \sum\limits_{i=1}^{m} M$ *is irreducible, i.e., there exists* $r > 0$ *such that* $H^r > 0$;

(iv) *there is a one-to-one correspondence of the* $\Delta \in \mathscr{F}_n$ *and the admissible* $J = (j_1 \cdots j_n)$ *such that*

$$\mu(\Delta) \approx \| M_{j_1} \cdots M_{j_n} \|.$$

We will consider the product of matrices in a more general setting. For a family of non-negative $d \times d$ matrices $\{M_1, \cdots, M_m\}$, we define the pressure function $P(q)$ by

$$P(q): = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|J|=n} \| M_J \|^q, \qquad q > 0. \tag{4.1}$$

The existence of the limit in the definition follows from a subadditive argument. The main theorem is

Theorem 4.3.   *Let* $M_1, \cdots, M_m$ *be non-negative* $d \times d$ *matrices such that* $H = \sum\limits_{i=1}^{n} M_i$ *is irreducible. Then the pressure function* $P(q)$ *is differentiable for* $q > 0$.

Combining Theorem 3.6, Theorems 4.2 and 4.3, we have the following conclusion

for the multifractal formalism.

Theorem 4. 4. *Let μ be the self-similar measure as in Theorem 4.2, then $\tau(q) = P(q)/\log\rho$, $q \in \mathbf{R}$. Consequently, $\tau(q)$ is differentiable for $q > 0$ and hence $f(\alpha) = \tau^*(q)$, $q > 0$ for $\alpha = \tau'(q)$.*

In the following we give a sketch of the proof of Theorem 4.3 together with some discussions. We need some standard notations: let $\Sigma = \{1, \cdots, m\}^{\mathbf{N}}$, $\Sigma_n = \{1, \cdots, m\}^n$ and $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma_n$; for $J \in \Sigma_n$, let $[J]$ denote the cylinder set with base J. We call a probability measure ν a Young measure if for ν-almost all $x \in \mathrm{supp}\,\nu$ the local dimensions $\alpha(x)$ exist and are equal. In this case the local dimension equals $\dim_1(\nu)$ ($= \dim_{\mathcal{H}}(\nu)$ (see Theorem 2.1). A sufficient condition for ν to be a Young measure is that ν is ergodic. The following differentiability result is due to Heurteaux[H].

Proposition 4. 5. *Let ν be a Young measure on Σ. Suppose there exists a constant $C > 0$ such that*

$$\nu([IJ]) \leqslant C\nu([I])\nu([J]), \qquad \forall\, I, J \in \Sigma^*. \tag{4.2}$$

Then $\tau_\nu(q)$ is differentiable at 1.

The derivative $\tau'_\nu(1)$ is the entropy of ν in Section 2. The proposition is a converse of a theorem of Ngai[N] where he proved that the existence of $\tau'_\nu(1)$ implies that ν is a Young measure.

The main objective is to use the matrices in Theorem 4.3 to construct a measure ν in Proposition 4.5. First we make use of Kolmogorov consistence theorem to define the measure

$$\tilde{\nu}([J]) = \lambda^{-n}\mathbf{u}'M_J\mathbf{v}, \quad J \in \Sigma_n,$$

where λ is the maximal eigenvalue of H, and $\mathbf{u}'$ and $\mathbf{v}$ are the corresponding eigenvectors of H and $\mathbf{u}'\mathbf{v} = 1$. It is direct to show that

$$\tilde{\nu}([J]) \approx \lambda^{-n} \| M_J \|, \quad J \in \Sigma_n, \tag{4.3}$$

and this implies that there exists $C > 0$ such that

$$\tilde{\nu}([IJ]) \leqslant C\tilde{\nu}([I])\tilde{\nu}([J]), \qquad \forall\, I, J \in \Sigma^*.$$

For $q \in \mathbf{R}$, let $s_n(q) = \sum_{|I|=n}\tilde{\nu}([I])^q$, where the summation is taken over all $I \in \Sigma_n$ with $\tilde{\nu}([I]) > 0$. By using the definition of

$$\tilde{\tau}(q) \; := \tau_{\tilde{\nu}}(q) = \lim_{n \to \infty}\frac{\log s_n(q)}{\log m^{-n}}$$

and (4.3), it is easy to see that

$$\bar{\tau}(q) = \frac{1}{\log m}(q\log\lambda - P(q)), \qquad \forall\ q > 0.$$

Therefore to prove Theorem 4. 3, it suffices to prove $\bar{\tau}(q)$ is differentiable. In order to apply Proposition 4. 5, we need to find yet another ν to have in addition the ergodic property (it will be a Young measure). For this we make use of a technique of Brown, Michon and Peyriere[BMP]. Let

$$\nu_{n,q}([J]) = \frac{\bar{\nu}([J])^q}{s_n(q)}, \qquad \forall\ J \in \Sigma_n,$$

let ν_q be a weak*-limit of the sequence, and let $\nu = \lim\limits_{n\to\infty}\frac{1}{n}(\nu_q + \nu_q \circ \sigma^{-1} + \cdots + \nu_q \circ \sigma^{-(n-1)})$ where σ is the natural shift operator on Σ. We can show that ν is ergodic and satisfies the inequality in (4. 2), hence $\tau_\nu(t)$ is differentiable for $t>0$. To conclude that $\tau_{\bar{\nu}}(q)$, $q>0$ is differentiable, we need to observe that

$$\tau_\nu(t) = \tau_{\bar{\nu}}(qt) - t\tau_{\bar{\nu}}(q), \qquad q > 0.$$

This completes the outline of the proof.

We remark that if we assume that the entries of M_i are strictly positive, then the above $\bar{\nu}$ is quasi-Bernoulli: there exists $C>0$ such that

$$C^{-1}\bar{\nu}([I])\bar{\nu}([J]) \leqslant \bar{\nu}([IJ]) \leqslant C\bar{\nu}([I])\bar{\nu}([J]), \qquad \forall\ I, J \in \Sigma^*.$$

We can use the similar proof to show that $P(q)$ is differentiable for $q<0$ as well. In general we do not have a conclusion for $q<0$. The examples in the next two sections show that $P(q)$ has non-differentiable points for $q<0$.

In view of the case of scaling functions, it will also be interesting to know Theorem 4. 3 for the M_i with negative entries. The reader can refer to [DL2], [LMW] for some special cases on the scaling function related to the pressure function defined by (4. 1).

Our investigation can actually be set up more generally in dynamical systems. Let (Σ_A, σ) be a subshift of finite type and let $M(x)$ be a Hölder continuous function on (Σ_A, σ) with values in the $d \times d$ non-negative matrices. We define the pressure function as

$$\mathscr{P}(q) = \lim_{n\to\infty}\frac{1}{n}\log \sum_{J \in \Sigma_{A,n}} \sup_{x \in [J]} \| M(x)\cdots M(\sigma^n x) \|^q,$$

where $\Sigma_{A,n}$ denotes the set of admissible indices of length n and $[J]$ denotes the cylinder set of $x \in \sum_A$. The pressure function of the scalar case (i. e. , $M(x) = e^{\phi(x)}$, where $\phi(x)$ is a real valued function called the potential of the subshift) has been studied in great detail in statistical mechanics and dynamical systems in conjunction with the Gibbs measure, the entropy and the variational principle[P], [Ru]. The investigation in [FL] is a natural exten-

sion of the classical case to the matrix—valued dynamical systems.

5   Erdös Measure

The IFS $S_1 x = \rho x$, $S_2 x = \rho x + (1-\rho)$ with $\rho = (\sqrt{5} - 1)/2$ satisfies the finite type condition and Theorem 4.4 holds. To construct the families of basic net intervals $\{\mathscr{F}_n\}_{n=1}^{\infty}$ and the matrices in Theorem 4.2, it is more convenient to employ a technique of Strichartz et al. [STZ] by reducing the overlapping of $S_1[0,1]$ and $S_2[0,1]$ into nonoverlapping sets. Let

$$T_0 x = S_1 S_1 x = \rho^2 x,$$

$$T_1 x = S_1 S_2 S_2 x = S_2 S_1 S_1 x = \rho^3 x + \rho^2,$$

$$T_2 x = S_2 S_2 x = \rho^2 x + \rho.$$

Then $T_0[0,1] = [0, \rho^2]$, $T_1[0,1] = [\rho^2, \rho]$, $T_2[0,1] = [\rho, 1]$ are three intervals with disjoint interiors. In terms of these maps, the self-similar identity (1.1) is reduced to three sets of second order identities: For $A \subseteq [0,1]$,

$$\begin{bmatrix} \mu(T_0 T_i A) \\ \mu(T_1 T_i A) \\ \mu(T_2 T_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0 A) \\ \mu(T_1 A) \\ \mu(T_2 A) \end{bmatrix}, \quad i = 0, 1, 2, \tag{5.1}$$

where

$$M_0 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

The identities (5.1) enables us to obtain the families of intervals $\{\mathscr{F}_n\}_{n=1}^{\infty}$ in Theorem 4.2; by Theorem 4.4, $\tau(q) = P(q)/\log\rho$.

For the explicit formula of $\tau(q)$, we can use recursive substitutions to express the product in terms of M_0 and M_2 and simplified: for any integer $k \geqslant 0$, let $J = (j_1, \cdots, j_k)$, $j_i = 0$ or 2,

$$\frac{1}{4}[0,1,0]M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2^{2k+3}} \| P_J \|,$$

where

$$P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The following theorem was proved in [LN1] for $q > 0$ and in [F1] for $q < 0$.

Theorem 5.1.   *Let $q_0 (\approx -2.25)$ be the root of $\sum\limits_{k=0}^{\infty} \sum\limits_{|J|=k} \| P_J \| = 1$.*

(i) *For $q \geqslant q_0$, $\tau(q)$ is given by α, the root of*

$$\sum_{k=0}^{\infty} \rho^{-(2k+3)\alpha} \Big(\sum_{|J|=k} c_J^q \Big) = 1. \tag{5.2}$$

(ii) *For $q < q_0$, $\tau(q) = \alpha_{q_0}$.*

This is a more complicated formula in comparison with (2.3) for the OSC. It follows that $\tau(q)$ is differentiable except at q_0; it is linear for $q \leqslant q_0$. By Theorem 4.4, the multifractal formalism holds for $q > 0$. For $q \in \mathbf{R}$, Feng and Olivier [FO] proved a general theorem that the formalism holds for the weak Gibbs measures μ. (In terms of the symbolic space (Σ, σ) with a shift σ, μ is a weak Gibbs measure means for any $\omega = (j_1, j_2, \cdots) \in \Sigma$,

$$\frac{1}{K(n)} \leqslant \frac{\mu([j_1, \cdots, j_{n-1}])}{\exp\left(-\sum\limits_{k=0}^{n-1} \phi(\sigma^k \omega)\right)} \leqslant K(n),$$

where ϕ is a non-negative function (potential) on Σ and $\lim\limits_{n} \log K(n)/n = 0$.) The following settles the problem of multifractal formalism for the Erdös measure.

Theorem 5.2.   *The Erdös measure is a weak Gibbs measure and $f(\alpha) = \tau^*(\alpha)$ for all $\alpha \in \mathrm{dom}\ (\tau^*)$.*

For the case that q is a nonnegative integer, we have algorithm to calculate $\tau(q)$ efficiently:

Theorem 5.3.   *For $q \in \mathbf{N}$, $q \geqslant 2$, (5.2) can be reduced to a polynomial equation $P(z) = 0$ (with $z = 2^q \rho^\alpha$). In this case, $\tau(q) = \log(z/2^q)/\log\rho$, where z is the largest positive root of P.*

By Theorem 5.2, we also conclude that

Theorem 5.4.   *The entropy dimension*

$$\dim_1(\mu) = \tau'(1) = \frac{1}{9\log\rho} \sum_{k=0}^{\infty} \sum_{|\sigma|=k} c_\sigma \log c_\sigma (\approx 0.9957).$$

There is also of interest to consider the more general Bernoulli convolution where the ρ^{-1} is a P. V. number. If the IFS satisfies the finite type condition, the multifractal formalism still hold for $q > 0$ according to Theorem 4.4 , However the above simple second

order identity does not work and the implementation of Theorem 4. 3 is more difficult, and we do not have a clear understanding for $q < 0$.

6   Convolution of Cantor Measure

Let ν be the standard Cantor measure, then ν is generated by the two maps

$$S_j(x) = \frac{1}{3}(x + 2j), j = 0, 1$$

with weights $\frac{1}{2}$ on each S_j. Its N-thconvolution $\mu = \nu * \cdots * \nu$ is generated by

$$S_j(x) = \frac{1}{3}(x + 2j),$$

with weights $\frac{1}{2^N}\begin{pmatrix} N \\ j \end{pmatrix}$, $j = 0, 1, \cdots, N$. It is well known that ν has only one local dimension, namely, $\log 2 / \log 3$. For $\mu = \nu * \nu$, the IFS $\{S_j\}_{j=0}^2$ satisfies the open set condition, there is an explicit formula for the L^q-spectrum (Theorem 2. 4)

$$\tau(q) = \left| \frac{\log\left(2(\frac{1}{4})^q + (\frac{1}{2})^q \right)}{\log 3} \right|$$

and the multifractal formalism holds. For the N-fold convolution, $N \geqslant 3$, the IFS $\{S_j\}_{j=0}^N$ satisfies the WSC but not the OSC. We consider the case $N = 3$ in particular. Let

$$\mu(\cdot) = \sum_{j=0}^{3} p_j \mu(3 \cdot - 2j). \tag{6.1}$$

For the three fold-convolution p_0, p_1, p_2, $p_3 = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ respectively. By using the technique of the dilation equation in Section 4, we can express μ into a vector-valued measure $\boldsymbol{\mu}$ on $\mathbf{R}$:

$$\boldsymbol{\mu}(A) = \begin{bmatrix} \mu(A \cap [0,1]) \\ \mu((A \cap [0,1]) + 1) \\ \mu((A \cap [0,1]) + 2) \end{bmatrix},$$

for any Borel subset $A \subset \mathbf{R}$. It is direct to check that supp$\mu \subset [0,1]$ and (6. 1) is equivalent to

$$\boldsymbol{\mu}(A) = \sum_{j=0}^{2} T_j \boldsymbol{\mu}(3A - j), \tag{6.2}$$

where

$$T_0 = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ p_3 & 0 & p_2 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & p_0 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_3 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} p_1 & 0 & p_0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}.$$

Let J be the interval $\left[\sum_{k=1}^{n} 3^{-k} j_k, \sum_{k=1}^{n} 3^{-k} j_k + 3^{-n} \right] \subset [0,1]$, then

$$\mu(J + i) = e_i^t T_J a, \qquad i = 0, 1, 2, \tag{6.3}$$

where $a = \mu([0,1]) = [a_0, a_1, a_2]^t$, and $\mathbf{e}_i$ is the unit vector in $\mathbf{R}^3$ whose $(i+1)$-th coordinate is 1. It can be proved that

$$\tau(q) = \lim_{n \to \infty} \frac{\log \sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_i^t T_J \mathbf{1})^q}{-n \log 3}. \tag{6.4}$$

Let $s_0 = p_1^q + p_2^q$, and for $n \geq 1$,

$$s_n: = s_n(q) = \sum_{J \in \{0,2\}^{n-1}} \left([p_2, p_1] P_J \begin{bmatrix} p_0 \\ p_3 \end{bmatrix} \right)^q;$$

$$b_n: = b_n(q) = \sum_{J \in \{0,2\}^{n-1}} ([p_2, p_1] P_J \mathbf{1})^q.$$

with

$$P_0 = \begin{bmatrix} p_0 & 0 \\ p_3 & p_2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} p_1 & p_0 \\ 0 & p_3 \end{bmatrix}.$$

Let $u_n^{(i)} = \sum_{|J|=n} (\mathbf{e}_i T_J \mathbf{1})^q$, $n \in \mathbf{N}$, $i = 1, 2, 3$. Then

$$\sum_{i=0}^{2} \sum_{|J|=n} (\mathbf{e}_{i-1}^t P_J \mathbf{1})^q = u_n^{(1)} + u_n^{(2)} + u_n^{(3)}$$

and

$$u_n^{(2)} = \sum_{k=0}^{n-1} s_k u_{(n-1)-k} + b_n, \qquad n \geq 1.$$

The last equation is the well known renewal equation and we can determine the growth rate of $u_n^{(2)}$:

$$\lim_{n \to \infty} (u_n^{(2)})^{1/n} = r^{-1},$$

where $\sum_{k=0}^{\infty} s_k r^{k+1} = 1$. Also it is not hard to show that

$$u_n^{(i)} \leqslant C \sum_{k=0}^{n} p_i^{kq} u_{n-k}, \qquad i = 0, 3.$$

By using these we have

Theorem 6.1. *Let $r: = r(q)$ be the unique solution such that $\sum_{k=0}^{\infty} s_k r^{k+1} = 1$. Then the*

L^q-spectrum $\tau(q)$ of μ in (6.4) is given by

$$\tau(q) = \frac{1}{\log 3} \min \{ -q \log p_0, -q \log p_3, \log r(q) \}, \quad q \in \mathbf{R}$$

and $\tau(q) = \log r(q) / \log 3$ if $q > 0$.

It is seen that $\tau(q)$ may have non-differentiable points, and indeed it happens in the case of 3-fold convolution of the Cantor measure. In this case the weight is $[p_0, p_1, p_2, p_3]$ $= [\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}]$ and

$$P_0 = 2^{-3} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \quad P_2 = 2^{-3} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}.$$

Theorem 6.2. *Let μ be the 3-fold convolution of the Cantor measure. Let $r(q)$ be the root of*

$$3^q \sum_{k=0}^{\infty} 2^{-kq} \Big(\sum_{|J|=n} \| P_J \|^q \Big) r^{k+1} = 1$$

and let $q_0 (\approx -1.149)$ be such that $r(q_0) = 2^{q_0}$. Then

$$\tau(q) = \begin{cases} \log r(q) / \log 3, & \text{if } q \geqslant q_0, \\ 3q \log 2 / \log 3, & \text{if } q < q_0. \end{cases}$$

The more striking phenomenon for the convolution of the Cantor measure, is the existence of an isolation point on the set of local dimensions $E = \{ \alpha: \alpha(s) = \alpha \text{ for some } s \in \text{supp} \mu \}$.

Theorem 6.3. *Let μ be the N-fold convolution of the Cantor measure $(N \geqslant 3)$. Then*

$$\bar{\alpha} = \sup \{ \bar{\alpha}(s): s \in \text{supp } \mu \} = \frac{N \log 2}{\log 3} \text{ is an isolated point of } E.$$

For the case $N = 3$, the result can be made more precise.

Theorem 6.4. *Let μ be the 3-fold convolution of the Cantor measure. Then*

(i) $\bar{\alpha} = \sup \{ \bar{\alpha}(s): s \in \text{supp} \mu \} = \dfrac{3 \log 2}{\log 3} \approx 1.89278;$

$\underline{\alpha} = \inf \{ \underline{\alpha}(s): s \in \text{supp} \mu \} = \dfrac{3 \log 2}{\log 3} - 1 \approx 0.89278.$

(ii) $E = [\underline{\alpha}, \tilde{\alpha}] \cup \{ \bar{\alpha} \}$ *with* $\tilde{\alpha} = \dfrac{3 \log 2}{\log 3} - \dfrac{\log b}{2 \log 3} \approx 1.1335$, *where* $b = \dfrac{7 + \sqrt{13}}{2}$.

The proof of the theorem is combinatoric [HL]. It depends on some careful counting of the multiple representation $s = \displaystyle\sum_{j=1}^{\infty} 3^{-j} x_j$, $x_j = 0, \cdots, N$ for $s \in \text{supp} \mu$. Since we can set up the measure locally in the matrix form as in (6.3), we can also make use of this to prove the above theorem [FLW].

In order for the multifractal formalism to hold, $f(\alpha)$ must be a concave function, so that the domain E must be an interval. This is true for all self-similar measures generated by the IFS satisfying the OSC. Theorem 6.3 implies that the multifractal formalism fails at $\bar{\alpha}$. Nevertheless, we see that the isolated point $\bar{\alpha}$ in E comes from the two end points in the support of the measure. If we exclude these two points, we have [FLW].

Theorem 6.5. *Let μ be as in Theorem 6.2. and let $\tilde{\mu}$ be the measure restricted on $K \subset$* (0, 3). *Then*

$$f_{\tilde{\mu}}(\alpha) = \tau^*(\alpha), \qquad \alpha \in (\underline{\alpha}, \alpha^*).$$

Recently Shmerkin [S] consider the more general case: $S_j(x) = \frac{1}{k}(x + b_j)$ where k, b_j $\in \mathbf{N}$, $j = 1, \cdots N$ with weights $\{p_0, \cdots, p_N\}$. He extended the two theorems to the case $N < 2k - 2$. Moreover he proved that, in the above notation

$$\underline{\alpha} = \limsup_{n \to \infty} \frac{\log(\max_{|J|=n} \| M_J \|)}{n \log k}$$

($\limsup_{n \to \infty}(\max_{|J|=n} \| M_J \|^{1/n})$ is the spectral radius of $\{M_1, \cdots, M_m\}$) and

$$\alpha^* = \inf_n \left\{ \frac{\log(\max_{|J|=n} \| M_J \|)}{n \log k} \right\}.$$

This gives a more complete answer to the problem.

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